Q1

[3]

$$i = \sqrt{-1} \implies i^2 = -1$$

$$z = a + bi$$

$a = \mathrm{Re}(z)$ $\quad$ $b = \mathrm{Im}(z)$

Note: Though z is a complex number, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are just 'normal' real numbers.

$z = a + bi \qquad z^* = a - bi$

z^* is the complex conjugate of z

Note: $z \times z^*$ is always a real number!

$$zz^* = (a+bi)(a-bi) = a^2 - b^2 i^2 = a^2 + b^2$$

$$\frac{1}{z}+\frac{1}{z^*}=\frac{z^*+z}{zz^*}$$

$z=x+iy \quad z^*=x-iy \quad z+z^*=2x$

$$zz^*=(x+iy)(x-iy)=x^2-\cancel{xiy}+\cancel{xiy}-i^2y^2$$

$$zz^*=x^2-i^2y^2=x^2-(-1)y^2=x^2+y^2$$

$$\frac{1}{z}+\frac{1}{z^*}=\frac{2x}{x^2+y^2}$$

(i) $\mathrm{Re}\left(\frac{1}{z}+\frac{1}{z^*}\right)=\mathrm{Re}\left(\frac{2x}{x^2+y^2}\right)=\frac{2x}{x^2+y^2}$

(ii) $\mathrm{Im}\left(\frac{1}{z}+\frac{1}{z^*}\right)=\mathrm{Im}\left(\frac{2x}{x^2+y^2}\right)=0$

Note: You could also do this by working out

$$\frac{1}{z}=\frac{1}{x+iy}\times\frac{x-iy}{x-iy}=\frac{x-iy}{x^2+y^2}$$

$$\frac{1}{z^*}=\frac{1}{x-iy}\times\frac{x+iy}{x+iy}=\frac{x+iy}{x^2+y^2}$$

and adding them together.

Q2

$z=a+bi \qquad z^*=a-bi$

z^* is the complex conjugate of z

Note: $z\times z^*$ is always a real number!

$$zz^*=(a+bi)(a-bi)=a^2-b^2i^2=a^2+b^2$$

$$\frac{2z_1}{z_2} = -1-7i \implies z_2 = \frac{2z_1}{-1-7i}$$

$$z_2 = \frac{2(3-4i)}{-1-7i} = \frac{6-8i}{-1-7i}$$

$$z_2 = \frac{6-8i}{-1-7i} \times \frac{-1+7i}{-1+7i}$$

$$= \frac{-6+42i+8i-56i^2}{1-\cancel{7i}+\cancel{7i}-49i^2}$$

$$= \frac{-6+50i-56(-1)}{1-49(-1)}$$

$$= \frac{50+50i}{50} = 1+i$$

So we have

$$\boxed{\begin{array}{c} z_2 = 1+i \\ (a=1,\ b=1) \end{array}}$$

Q3

Find all the complex numbers z for which $z^2 = 4z^*$.

[4]

Important fact

Two complex numbers are equal if and only if both their real and imaginary parts are equal.

$z = x+iy \quad z^* = x-iy \quad x, y \in \mathbb{R}$

$z^2 = (x+iy)^2 = x^2+2xyi+i^2y^2 = (x^2-y^2)+(2xy)i$

$4z^* = 4x-4yi$

So $(x^2-y^2)+(2xy)i = 4x-4yi$, which means that

$x^2-y^2 = 4x$ ① ← simultaneous equations

and $2xy = -4y \implies xy+2y = y(x+2) = 0$ ②

Equation ② only has solutions $y=0$ and $x=-2$.

Looking now at equation ①:

If $y=0$: $x^2 = 4x \implies x=0$ or $x=4$

If $x=-2$: $4-y^2=-8 \implies y^2=12 \implies y=\pm 2\sqrt{3}$

So the solutions are:

$x=0, y=0 \quad x=4, y=0$

$x=-2, y=2\sqrt{3} \quad x=-2, y=-2\sqrt{3}$

Therefore $z^2=4z^*$ when

$$\boxed{z = 0,\ 4,\ -2+2\sqrt{3}i,\ -2-2\sqrt{3}i}$$

Note: This question can also be answered by expressing z in modulus-argument or exponential form and solving $z^2=4z^*$ in that form. Done correctly, it will give the same 4 answers.

Q4

Since $\alpha = 3$ is a root, then $(z-3)$ is a factor of $z^3 + (b-3)z^2 + (7-3b)z - 21$.

Therefore

$$z^3 + (b-3)z^2 + (7-3b)z - 21 = (z-3)(z^2 + bz + 7) = 0$$

$z^2 + bz + 7$ will have non-real roots if its discriminant is negative:

$$b^2 - 4(1)(7) < 0$$

$$b^2 - 28 < 0$$

$$b^2 < 28$$

$$-\sqrt{28} < b < \sqrt{28}$$

$$\boxed{-2\sqrt{7} < b < 2\sqrt{7}}$$

Q5

Given that $3 + qi$ is one of the roots of the quadratic equation $z^2 - 12pz + 58 = 0$, where p and q are positive real constants, find the values of p and q.

[4]

Important fact

In quadratic, cubic, and quartic equations with real coefficients, non-real complex roots only occur in complex conjugate pairs.

$(3+qi)^* = 3 - qi$ is another root.

Therefore

$$\begin{aligned} z^2 - 12pz + 58 &= (z-(3+qi))(z-(3-qi)) \\ &= z^2 - (3 - \cancel{qi} + 3 + \cancel{qi})z + (3+qi)(3-qi) \\ &= z^2 - 6z + (9 - \cancel{3qi} + \cancel{3qi} - q^2 i^2) \end{aligned}$$

$$z^2 - 12pz + 58 = z^2 - 6z + (9 + q^2)$$

It follows that

$$-12p = -6 \Rightarrow p = \frac{1}{2}$$

and $9 + q^2 = 58 \Rightarrow q^2 = 49 \Rightarrow q = \pm 7$

But $q > 0$ so $q = 7$.

The solutions are

$$\boxed{p = \frac{1}{2} \quad q = 7}$$

Q6

Complete the square

$$z^2 - 6iz - 14 = (z-3i)^2 - 9i^2 - 14$$
$$= (z-3i)^2 + 9 - 14$$
$$= (z-3i)^2 - 5$$

So

$$(z-3i)^2 - 5 = 0$$
$$(z-3i)^2 = 5$$
$$z - 3i = \pm\sqrt{5}$$
$$z = 3i \pm \sqrt{5}$$

The solutions are

$$\boxed{\sqrt{5} + 3i \quad \text{and} \quad -\sqrt{5} + 3i}$$

Not complex conjugates

<u>Important fact</u>

In quadratic, cubic, and quartic equations <u>with real coefficients</u>, non-real complex roots only occur in complex conjugate pairs.

<u>Note</u>: This fact doesn't apply here, because the z coefficient $(-6i)$ is not a real number!

Q7a

a) $\alpha^2 = (1-3i)^2 = 1 - 6i + 9i^2 = -8 - 6i$

$\alpha^3 = \alpha^2\alpha = (-8-6i)(1-3i) = -8 + 24i - 6i + 18i^2 = -26 + 18i$

Because α is a solution,

$(-26+18i) + 4(-8-6i) + k(1-3i) + m = 0$

$-26 + 18i - 32 - 24i + k - 3ki + m = 0$

$(-58 + k + m) + (-6 - 3k)i = 0$

This can only be true if

Simultaneous equations

$-58 + k + m = 0 \Rightarrow k + m = 58$ ①

and $-6 - 3k = 0 \Rightarrow 3k = -6 \Rightarrow k = -2$ ②

Substituting ② into ①

$-2 + m = 58 \Rightarrow m = 60$

The values of k and m are

$\boxed{k = -2 \quad m = 60}$

Q7b

You are given that the complex number $\alpha = 1 - 3i$ satisfies the cubic equation

$$z^3 + 4z^2 + kz + m = 0,$$

where k and m are real constants.

(a) By first calculating α^2 and α^3, find the values of k and m. [5]

(b) Find the other two roots of the cubic equation. [3]

Important fact

In quadratic, cubic, and quartic equations with real coefficients, non-real complex roots only occur in complex conjugate pairs.

From part (a), $k = -2$ and $m = 60$

b) $(1-3i)^* = 1 + 3i$ is also root.

Therefore

$(z - (1-3i))(z - (1+3i)) = z^2 - 2z + 10$

is a factor of $z^3 + 4z^2 - 2z + 60$.

So

$z^3 + 4z^2 - 2z + 60 = (z^2 - 2z + 10)(z + 6) = 0$

$\Rightarrow z = -6$ is a root.

The other two roots of the equation are

$\boxed{1 + 3i \text{ and } -6}$

Q8a

$f(z) = z^4 - 2z^3 + 6z^2 + pz + 125$, where p is a real constant.

(a) Given that $3 - 4i$ is a root of the equation $f(z) = 0$, show that $z^2 + 4z + 5$ is a factor of $f(z)$.

[4]

(b) Hence find the value of p and solve completely the equation $f(z) = 0$.

[3]

Important fact

In quadratic, cubic, and quartic equations with real coefficients, non-real complex roots only occur in complex conjugate pairs.

a) $(3-4i)^* = 3+4i$ is another root.

Therefore $(z-(3-4i))(z-(3+4i)) = z^2 - 6z + 25$ is a factor of $f(z)$.

So

$f(z) = (z^2 - 6z + 25)(z^2 + bz + c) = z^4 - 2z^3 + 6z^2 + pz + 125$

$(z^2 - 6z + 25)(z^2 + bz + c)$

$= z^4 + (b-6)z^3 + (c - 6b + 25)z^2 + (25b - 6c)z + 25c$

Comparing z^3 and constant coefficients:

$b - 6 = -2 \Rightarrow b = 4$

$25c = 125 \Rightarrow c = 5$

Therefore

$f(z) = (z^2 - 6z + 25)(z^2 + 4z + 5)$

so

$z^2 + 4z + 5$ is a factor of $f(z)$.

Q8b

b) Comparing z coefficients from the part (a) working:

$p = 25b - 6c = 25(4) - 6(5) = 70$

$\boxed{p = 70}$

$3-4i$ and $3+4i$ are roots (part (a)).

The other roots are the solutions to

$z^2 + 4z + 5 = 0$

$(z+2)^2 - 4 + 5 = 0$

$(z+2)^2 = -1$

$z + 2 = \pm\sqrt{-1} = \pm i$

$z = -2 \pm i$

So $-2+i$ and $-2-i$ are the other roots

The solutions to $f(z) = 0$ are

$\boxed{z = 3-4i,\ 3+4i,\ -2+i,\ -2-i}$

Q9a

sen such
$x - iy$.

You could also get this by completing the square

[5]

why y
(z).

[2]

perties

[2]

[3]

a) Let $\text{Re}(z) = a$, $\text{Im}(z) = b$, so $z = a + bi$

Then $z = (\sqrt{z})^2 = (x+iy)^2 = (x^2 - y^2) + (2xy)i = a + bi$

This can only be true if [See fact on next page]

$$x^2 - y^2 = a \quad (1)$$

and $2xy = b \Rightarrow y = \frac{b}{2x}$ (2)

Substitute (2) into (1):

$$x^2 - \left(\frac{b}{2x}\right)^2 = a \quad \Rightarrow \quad 4x^4 - 4ax^2 - b^2 = 0$$

By the quadratic formula:

$$x^2 = \frac{4a \pm \sqrt{16a^2 + 16b^2}}{8} = \frac{a \pm \sqrt{a^2+b^2}}{2}$$

But x is a real number, so $x^2 \geq 0$. Therefore since $\sqrt{a^2+b^2} \geq |a|$, we must reject the 'minus' version. [See note on next page]

Then taking the square root and substituting $a = \text{Re}(z)$ and $b = \text{Im}(z)$ gives:

$$x = \sqrt{\frac{\text{Re}(z) + \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2}}{2}}$$

where $x \geq 0$ as required.

Important fact

Two complex numbers are equal if and only if both their real and imaginary parts are equal.

Note on rejecting the 'minus' version of the x^2 equation

Define $F_p = \frac{a + \sqrt{a^2+b^2}}{2}$ and $F_m = \frac{a - \sqrt{a^2+b^2}}{2}$

If $b \neq 0$: $F_p > 0$, $F_m < 0$, so rejecting F_m is obvious.

If $b = 0, a = 0$: $F_m = F_p = 0$, so rejecting F_m doesn't change anything. (In this case, $z = 0$, so $\sqrt{z} = 0$)

If $b = 0, a < 0$: $F_p = 0$, $F_m < 0$, so rejecting F_m is obvious. (In this case, z is a negative real number, and $\sqrt{z} = \sqrt{a}\, i$)

If $b = 0, a > 0$: $F_p = a$, $F_m = 0$. But if $x^2 = F_m = 0$, then $x^2 - y^2 = a$ gives $y^2 = -a < 0$. But $y \in \mathbb{R}$, so y^2 can't be negative. So we have to reject F_m. (In this case, z is a positive real number, and $\sqrt{z} = \sqrt{a}$)

b) From part (a), $2xy = \text{Im}(z)$, so

$$y = \frac{\text{Im}(z)}{2x}$$

Because $x > 0$, it follows that:

If $\text{Im}(z) > 0$, then $y > 0$

If $\text{Im}(z) = 0$, then $y = 0$

If $\text{Im}(z) < 0$, then $y < 0$

Q9C

The principal square root of a complex number z is defined as

$$\sqrt{z} = x + \mathrm{i}y$$

where x and y are real numbers and $x \geq 0$. If $x = 0$ then the value for y is chosen such that $y \geq 0$. Note that the other square root of z will then be given by $-\sqrt{z} = -x - \mathrm{i}y$.

(a) Show that

$$x = \sqrt{\frac{\text{Re}(z) + \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2}}{2}}$$

[5]

(b) Given that $x > 0$, derive a formula for y in terms of x and $\text{Im}(z)$, and explain why y in this case will always have the same sign (positive, negative, or zero) as $\text{Im}(z)$.

[2]

(c) Hence show that in general

$$y = \pm\sqrt{\frac{-\text{Re}(z) + \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2}}{2}}$$

with the choice of the positive or negative value being dependent on the properties of z.

[2]

(d) Explain what must be true of z for each of the following to be true:

(i) $x = 0, y \neq 0$

(ii) $x \neq 0, y = 0$

(iii) $x = 0, y = 0$

[3]

You can get the same result starting from $x^2 - y^2 = b \Rightarrow y^2 = x^2 - b$

c) Using $a = \text{Re}(z)$ and $b = \text{Im}(z)$, then from part (b):

$$y = \frac{b}{2x} \Rightarrow y^2 = \frac{b^2}{4x^2}$$

So $$y^2 = \frac{b^2}{2(a + \sqrt{a^2+b^2})}$$

$$y^2 = \frac{b^2}{2(a+\sqrt{a^2+b^2})} \times \frac{a - \sqrt{a^2+b^2}}{a - \sqrt{a^2+b^2}}$$ Rationalise the denominator

$$y^2 = \frac{b^2(a - \sqrt{a^2+b^2})}{2(a^2 - (a^2+b^2))} = \frac{\cancel{b^2}(a - \sqrt{a^2+b^2})}{-2\cancel{b^2}}$$

$$y^2 = \frac{-a + \sqrt{a^2+b^2}}{2}$$

Taking the square root and substituting for a and b:

$$y = \pm\sqrt{\frac{-\text{Re}(z) + \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2}}{2}}$$

where

If $x > 0$, use 'plus' version if $\text{Im}(z) \geq 0$ and 'minus' version if $\text{Im}(z) < 0$.

If $x = 0$, choose 'plus' version so that $y \geq 0$.

Q9d

d) (i) If $x=0$, $y\neq 0$, then

$\sqrt{z} = iy \Rightarrow z = (iy)^2 = i^2 y^2 = -y^2$

$\boxed{z \text{ is a negative real number}}$

(ii) If $x\neq 0$, $y=0$, then

$\sqrt{z} = x \Longrightarrow z = x^2$

$\boxed{z \text{ is a positive real number}}$

(iii) If $x=0$, $y=0$, then

$\sqrt{z} = 0 \Longrightarrow z = 0$

$\boxed{z \text{ is equal to zero}}$